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ON DIFFERENTIAL PROPERTIES OF MULTIVARIABLE FUNCTIONS

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The Disertation will be available in the Akaki Tsereteli State University Library (59 Tamar Mepe St., Kutaisi, 4600)

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Topicality of Research. Studies of differential properties of one-variable and multi-variable functions represent one of the main issues of real analysis, which features different aspects on what kind of function or what kind of differentiation is examined. In this respect, particular emphasis in studies has been placed on the function classes of bounded variation. A classical result of Lebesgue on differentiability of one-variable function of bounded variation almost everywhere, became an initial point of actual research conducted by Lebesgue, Saks, Busemann and Feller, Zygmund, Burkill and Haslam-Jones, Stepanov, Kronrod, Vitushkin and others, and they were aimed at multidimensional extension of the mentioned result of Lebesgue through the different possible settings, or even at impossibility of establishing the extension. The picture of the obtained results along this line at the moment can be developed by monographs of Saks [Sa1], de Guzman [Gu], Vitushkin [Vi], Oniani [O1] and works of Zerekidze [Ze] and Stokolos [Sk].

There are different definitions of differentiation (derivation) (for example, see the [Sa1], [Gu],[Sn],[Br],[Mu]), which represent the variations of two classical definitions: the first one – is an ordinary differentiation (that means possibility of local approximation of a function by the a linear mapping); another one – is a derivative in the Bettazzi sense which is a limit of a ratio of mixed difference of a function on n-dimensional interval (or value of a function of interval) to the volume of the same interval. Generalization of classical definition of gradient was suggested by Dzagnidze [Dz1] and then by its application it was established a number of interesting structural properties of multivariable functions. Along this line, the obtained results are gathered in monographs [Dz2] and [Dz3].

The Aim of Dissertation. The Study of divergence characteristics of the strong means of additive functions of intervals having bounded variation; investigation of differential properties of multivariable functions having bounded variation in the Hardy and Arzela sense, from the standpoint of the existence of strong gradient; comparing the condition of existence of generalized gradient with respect to a basis with the condition of differentiability point-wise and on sets of positive measure.

Research Novelty.

1) There is obtained an exact estimate of the rate of divergence of strong means of an additive function of intervals having bounded variation;
2) There is established that each function having bounded variation in the Hardy sense has a strong gradient at almost every point (that is a stronger conclusion then the one on differentiability almost everywhere);
3) There is constructed an example of continuous function having bounded variation in the Arzela sense, which almost nowhere has a strong gradient;
4) There is introduced the definition of generalized gradient with respect to a basis and is made a complete solution of question of comparing the condition of existence of generalized gradient with respect to a basis and the condition of differentiability;
5) There are found the following two natural properties: completeness and anisotropic density, with which the existence of generalized gradient with respect to a given basis is an essentially stronger condition than differentiability.

Approbation of Work. The dissertation results have been presented at the Scientific seminar on Function Theory and Functional Analysis at the Ivane Javakhishvili Tbilisi State University (Head of the Seminar – Academician L. Zhizhiashvili), at the Scientific Seminar of the Department of Mathematics at the Akaki Tsereteli State University (Head of the seminar – Professor G.G. Oniani), at the II Scientific Conference of Young Scientists (Kutaisi, October 8-10, 2004) and at the III International Conference of Georgian Mathematical Union (Batumi, September 2-9, 2012).

Publication. There are published five scientific works, which are listed below the text of this author's abstract.

The size and the structure. The dissertation contains 76 pages. It consists of the introduction, two chapters and bibliography. The bibliography contains 37 names.

Content of Dissertation

In the first chapter there are considered the differential properties of the following three types of functions: the additive functions of an interval having bounded variation; the functions of bounded variation in the Hardy sense; the functions of bounded variation in the Arzela sense.

In the first and second paragraphs there is provided the necessary information on multivariable functions having the different types of bounded variation.

In the third paragraph there is obtained an exact estimate of the rate of divergence of strong means of an additive function of intervals having bounded variation.

Let F be a function defined on the class of all n-dimensional intervals (briefly: function of intervals). F is said to be an additive if for any non-overlapping intervals $I_1,\dots,I_m$ whose union is an interval I we have

$$F(I)=\sum_{k=1}^{m}F(I_k).$$

A partition of an interval I is a finite family of non-overlapping intervals whose union is I. Let Π_I be the collection of all partitions of I.

A function of intervals F is said to be of bounded variation on an interval I if

$$V_I(F)=\sup_{P\in\Pi_I}\sum_{J\in P}|F(J)|<\infty.$$

F is said to be of bounded variation on $\mathbb{R}^n$ if the supremum of $V_I(F)$ taken over all intervals I is finite. Denote the supremum by $V(F)$. The class of all additive functions of n-dimensional intervals of bounded variation on $\mathbb{R}^n$ denote by $V(\mathbb{R}^n)$.

For $x \in \mathbb{R}^n$, let us denote by $\mathbb{I}(x)$ the family of all n-dimensional intervals $I = [a_1,b_1]\times\cdots\times[a_n,b_n]$ with $b_k - a_k > 0$ $(k \in \overline{1,n})$ containing x.

For n -dimensional interval $I = [a_1,b_1]\times\cdots\times[a_n,b_n]$ denote $l_k(I) = b_k - a_k$ $(k \in \overline{1,n})$.

A function $w:(0,1)^{n-1} \to (0,\infty)$ which is decreasing with respect to each variable will be called a weight. We will say that a weight w satisfies the condition (K) if

$$\int_{(0,1)^{n-1}} \frac{dt_1\cdots dt_{n-1}}{t_1\cdots t_{n-1} w(t_1,\ldots,t_{n-1})} < \infty.$$

Remark 1.3.1. Condition (K) is satisfied by functions

$$\left(\ln\frac{1}{t_1}\cdot\ln\frac{1}{t_2}\cdot\dots\cdot\ln\frac{1}{t_{n-1}}\right)^{1+\varepsilon} \qquad (n\geq 2,\ \varepsilon>0)$$

and is not satisfied by the function

$$\ln\frac{1}{t_1}\cdot\ln\frac{1}{t_2}\cdot\dots\cdot\ln\frac{1}{t_{n-1}}.$$

The same may be said about more general type functions, such as

$$w_{n,k,\varepsilon}(t_1,\cdots,t_{n-1}) = \prod_{i=1}^{n-1}\left[\ln\frac{1}{t_i}\cdot\ln\ln\frac{1}{t_i}\cdot\dots\cdot\left(\overbrace{\ln\ln\cdots\ln}^{k-times}\frac{1}{t_i}\right)^{1+\varepsilon}\right],$$

when $\varepsilon > 0$ and $\varepsilon = 0$, accordingly. Thus, when $n = 2$, condition (K) is satisfied by functions $\left(\ln\frac{1}{t}\right)^{1+\varepsilon}$ $(\varepsilon > 0)$, and is not satisfied by the function $\ln\frac{1}{t}$.

Let P_n be the class of all permutations of the set $\{1,\ldots,n\}$.

For a weight w, $f \in L(\mathbb{R}^n), F \in V(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ denote

$$M_w(f)(x) = \sup_{I\in\mathbb{I}(x), diam I<1}\ \max_{p\in P_n} \frac{1}{|I|\, w(l_{p(1)}(I),\ldots,l_{p(n-1)}(I))}\int_I |f|;$$

$$T_w(F)(x) = \sup_{I\in\mathbb{I}(x), diam I<1}\ \max_{p\in P_n} \frac{V_I(F)}{|I|\, w(l_{p(1)}(I),\ldots,l_{p(n-1)}(I))}.$$

We denote by $c(\alpha_1,\ldots,\alpha_m)$ positive constants depending only on the parameters $\alpha_1,\ldots,\alpha_m$.

S. Saks [Sa2] and H. Busemann and W. Feller [BF] constructed a function $f \in L(\mathbb{R}^n), f \geq 0$, whose strong integral means diverge unboundedly at every point, in particular,

$$\limsup_{I\in\mathbb{I}(x), diam I\to 0} \frac{1}{|I|}\int_I f = \infty$$

for every $x \in \mathbb{R}^n$.

Note that the maximal integral class, where the convergence of strong means is guaranteed, is $L(1+\ln^+ L)^{n-1}(\mathbb{R}^n)$ (see [JMZ] and [Sa3]).

G. Karagulyan [Ka] obtained an exact estimate of the rate of divergence of strong integral means. Namely, he proved

Theorem 1.3.A. *If a weight w satisfies the condition* (K), *then for every $f \in L(\mathbb{R}^n)$*

$$\frac{1}{|I|}\int_I f = o[\min_{p\in P_n} w(l_{p(1)}(I),\ldots,l_{p(n-1)}(I))],\quad as\quad I\in\mathbb{I}(x),\ diam I\to 0$$

for almost every $x \in \mathbb{R}^n$; furthermore the operator M_w is of weak type $(1,1)$, *i.e.,*

$$|\{M_w(f) > \lambda\}| \leq \frac{c(w,n)}{\lambda}\int_{\mathbb{R}^n}|f| \quad (f\in L(\mathbb{R}^n),\ \lambda>0).$$

If a weight w *does not satisfy the condition* (K), *then there exists* $f \in L(\mathbb{R}^n)$, $f \geq 0$, *such that*

$$\limsup_{I \in \mathbb{I}(x), diam I \to 0} \frac{1}{|I| \, w(l_1(I), \dots, l_{n-1}(I))} \int_I f = \infty$$

for every $x \in \mathbb{R}^n$.

The following result extends Theorem 1.3.A to the strong means of additive functions of intervals with bounded variation.

Theorem 1.3.1. *If a weight* w *satisfies the condition* (K), *then for every* $F \in V(\mathbb{R}^n)$

$$\frac{F(I)}{|I|} = o[\min_{p \in P_n} w(l_{p(1)}(I), \dots, l_{p(n-1)}(I))], \quad as \quad I \in \mathbb{I}(x), \; diam I \to 0$$

for almost every $x \in \mathbb{R}^n$; *furthermore*

$$|\{T_w(F) > \lambda\}| \leq \frac{c(w,n)}{\lambda} V(F) \quad (F \in V(\mathbb{R}^n), \; \lambda > 0).$$

Remark 1.3.3. *Let* w *be a weight which does not satisfy condition* (K) *and let* f *be a function from the second part of Theorem 1.3.A corresponding to* w. *Then assuming* $F(I) = \int_I f$ *for any* n *-dimensional interval* I, *we conclude the exactness of the estimation given in Theorem* 1.3.1.

Remark 1.3.4. Theorem 1.3.1 finds its essential application in fourth paragraph, when studying the differential properties of functions having bounded variation in the Hardy sense.

In the fourth and fifth paragraphs there is established that a function of bounded variation in the Hardy sense almost everywhere has a strong gradient, but functions of bounded variation in the Arzela sense do not have a similar property.

Let us remind the necessary definitions and information on functions of bounded variation in the Hardy and in the Arzela sence, as well as on definition of the strong gradient.

For $x, y \in \mathbb{R}^n$ with $x \leq y$ (i.e., $x_i \leq y_i$ for every $i \in \overline{1,n}$), denote by I_x^y the interval $\prod_{i=1}^n [x_i, y_i]$. The mixed difference of $f : [0,1]^n \to \mathbb{R}$ on an interval $I = I_x^y \subset [0,1]^n$ is the quantity

$$\Delta(f, I) = \sum_{\varepsilon_1=0}^{1} \cdots \sum_{\varepsilon_n=0}^{1} (-1)^{n - \sum_{i=1}^n \varepsilon_i} f(x_1 + \varepsilon_1(y_1 - x_1), \dots, x_n + \varepsilon_n(y_n - x_n)).$$

Let Π be a family of all partitions of $[0,1]^n$.

A function $f : [0,1]^n \to \mathbb{R}$ is said to be of bounded variation in the Vitali sense if

$$\sup_{P \in \Pi} \sum_{I \in P} |\Delta(f, I)| < \infty.$$

Denote by $\mathbb{V}_n$ the class of all functions on $[0,1]^n$ of bounded variation in the Vitali sense.

Denote by $|B|$ a number of elements of a set $B \subset \overline{1,n}$.

For $B \subset \overline{1,n}$ with $0 < |B| < n, t \in [0,1]^{n-|B|}$ and $\tau \in [0,1]^{|B|}$, denote by (t, τ, B) the point of $\mathbb{R}^n$ for which $(t, \tau, B)_i = t_{|\overline{1,i} \setminus B|}$ if $i \notin B$ and $(t, \tau, B)_i = \tau_{|\overline{1,i} \cap B|}$ if $i \in B$.

Let f be a function on $[0,1]^n$. For $B \subset \overline{1,n}$ with $0 < |B| < n$ and $t \in [0,1]^{n-|B|}$, denote by $f_{B,t}$ the function on $[0,1]^{|B|}$ for which

$$f_{B,t}(\tau) = f((t, \tau, B)) \quad (\tau \in [0,1]^{|B|}).$$

Furthermore, denote $f_B = f_{B,0}$ where 0 is the zero element of $\mathbb{R}^{n-|B|}$. It is obvious that $f_{\overline{1,n}} = f$.

A function $f:[0,1]^n \to \mathbb{R}$ is said to be of bounded variation in the Hardy sense if f and its every section is of bounded variation in the Vitali sense, i.e., $f \in \mathbb{V}_n$ and $f_{B,t} \in \mathbb{V}_{|B|}$ for every $B \subset \overline{1,n}$ with $0 < |B| < n$ and $t \in [0,1]^{n-|B|}$. The class of all functions on $[0,1]^n$ with bounded variation in the Hardy sense is denoted by $\mathbb{H}_n$. By one result due to Leonov [Le],

$$f \in \mathbb{H}_n \Leftrightarrow f_B \in \mathbb{V}_{|B|} \quad (B \subset \overline{1,n}, B \neq \varnothing).$$

Recall that a Lebesgue indefinite integral of a function $f \in L[0,1]^n$ is defined as follows

$$F_f(x) = \int_{[0,x_1]\times\cdots\times[0,x_n]} f(y)dy \qquad (x \in [0,1]^n).$$

From the above-mentioned result of Leonov it obviously follows that a Lebesgue indefinite integral of an arbitrary function $f \in L[0,1]^n$ is of bounded variation in the Hardy sense.

A function $f:[0,1]^n \to \mathbb{R}$ is said to be increasing if $\Delta(f, I_x^y) \geq 0$ for any $x, y \in [0,1]^n$ with $x \leq y$. Denote by $\mathbb{M}_n$ the class of all increasing functions $f:[0,1]^n \to \mathbb{R}$. It is obvious that $\mathbb{M}_n \subset \mathbb{H}_n$. An increasing function is a.e. continuous (see [YY]) and therefore is measurable.

It is well known that a function of bounded variation in the Hardy sense can be decomposed as a difference of two increasing functions (see e.g.[AC]).

A function $f:[0,1]^n \to \mathbb{R}$ is said to be of bounded variation in the Arzela sense if the set of all sums

$$\sum_{k=1}^{m-1} |f(x_{k+1}) - f(x_k)|,$$

where $m \in \mathbb{N}$ and $(0,\ldots,0) = x_1 \leq x_2 \leq \cdots \leq x_m = (1,\ldots,1)$, is bounded. Note that every function of bounded variation in the Hardy sense is also of bounded variation in the Arzela sense (see e.g. [AC] or [Ho]).

For $n \geq 2, h \in \mathbb{R}^n$ and $i \in \overline{1,n}$, denote by $h(i)$ a point in $\mathbb{R}^n$ such that $h(i)_j = h_j$ for every $j \in \overline{1,n} \setminus \{i\}$ and $h(i)_i = 0$. For $i \in \overline{1,n}$, let L_i be a hyperplane $\{h \in \mathbb{R} : h_i = 0\}$.

Let f be a function defined in a neighborhood of a point $x \in \mathbb{R}^n$. If for $i \in \overline{1,n}$ there exists the limit

$$\lim_{\mathbb{R}^n \setminus L_i \ni h \to 0} \frac{f(x+h) - f(x+h(i))}{h_i}$$

then we call its value the i-th strong partial derivative of f at x and denote it by $D_{[i]}f(x)$. If f has finite $D_{[i]}f(x)$ for every $i \in \overline{1,n}$, then following Dzagnidze [Dz1] we say that there exists a strong gradient of f at x or f has a strong gradient at x.

The definition of strong gradient was introduced by O. Dzagnidze [Dz1] due to studies of differential properties of Lebesgue multiple indefinite integral.

As is known, the existence of ordinary gradients, i.e. of ordinary partial derivatives at the point, does not imply differentiability of a function. As distinguished from ordinary gradient, the strong gradient originates a stronger condition than differentiability is. In particular, O. Dzagnidze [Dz1] has established that: if the function has a strong gradient at a point, then it is differentiable at the same point; but the reverse implication is not generally valid, namely the function $f(x_1, x_2) = |x_1 x_2|^{2/3}$ is differentiable at point 0, but it does not have a strong gradient at the same point.

Let us say that a condition (A) is essentially stronger than a condition (B) (or (B) is essentially weaker than (A)) if: 1) The satisfaction of (A) at the fixed point implies the satisfaction of (B) at the same point; and 2) There is a function, for which (B) is satisfied at each point of a some set of positive measure, but (A) is not satisfied at no point of the same set.

G.G. Oniani [O2] established that the condition of existence of a strong gradient is essentially stronger than the condition of differentiability. In particular, he proved the following result.

Theorem 1.2.A. *For arbitrary* $n \geq 2$ *there exists a continuous function* $f:[0,1]^n \to \mathbb{R}$ *such that:*

1. f *is differentiable almost everywher ,*

2. $\overline{\lim\limits_{\mathbb{R}^n \setminus L_1 \ni h \to 0}} \dfrac{f(x+h)-f(x+h(1))}{h_1} = \infty$ *almost everywhere and consequantly,* f *almost nowhere has a strong gradient.*

Now, let us move to the review of results established in the fourth and fifth paragraphs.

The differential properties of multivariable functions having bounded variation were studied by different authors. In particular, Burkill and Haslam-Jones [BH] proved the following:

Theorem 1.4.A. *Each function* $f:[0,1]^n \to \mathbb{R}$ *of bounded variation in the Arzela sense (and consequently, in the Hardy sense) is differentiable almost everywhere.*

Besides, it is also known that every function on $[0,1]^n$ of bounded variation in the Kronrod-Vitushkin sense is differentiable a. e. (Kronrod [Kr] (for $n=2$), Vitushkin [Vi] (for arbitrary $n \geq 2$)). There exists a function on $[0,1]^2$ with bounded variation in the Tonelli sense which is nondifferentiable everywhere (Stepanoff [St]). Note that an analogous statement for functions of bounded variation in the Vitali sense is obvious.

Since an indefinite integral of arbitrary $f \in L[0,1]^n$ has a bounded variation in the Hardy sense, by virtue of Theorem 1.4.A it is differentiable a.e. However, as shown in Dzagnidze [Dz1] (for $n=2$) and Dzagnidze and Oniani

[DO] (for arbitrary $n \geq 2$), an indefinite integral has a stronger differential property, namely, it has a strong gradient a. e. In this context, there naturally arises a question whether an analogous conclusion is true for every function of bounded variation in the Hardy sense (in the Arzela sense) .

An answer to this question is provided by the following theorems.

Theorem 1.4.1. *Every function* $f:[0,1]^n \to \mathbb{R}$ *of bounded variation in the Hardy sense has a strong gradient almost everywhere.*

Remark 1.4.2. Theorem 1.3.1 is essentially used in the proof of Theorem 1.4.1.

Theorem 1.5.1. *For arbitrary* $n \geq 2$ *there exists a continuous function* $f:[0,1]^n \to \mathbb{R}$ *of bounded variation in the Arzela sense that nowhere has a strong gradient.*

If we take into account that evevry function of bounded variation in the Arzela sense is differentiable almost everywhere (see Theorem 1.4.A), we will derive from Theorem 1.5.1 the following improvement of Theorem 1.2.A.

Theorem 1.5.2. *For arbitrary* $n \geq 2$ *there exists a continuous function* $f:[0,1]^n \to \mathbb{R}$ *that is differentiable almost everywhere, but nowhere has a strong gradient.*

In the second chapter a question of comparing the conditions of existence of generalized gradient with respect to a basis and of differentiability there is considered.

In the first paragraph there is introduced the definition of generalized gradient with respect to a basis and reviewed the known results and questions related to this definition.

By Π_i $\left(i \in \overline{1,n}\right)$ denote the class of all sets $\Delta \subset \mathbb{R}^n$ with the properties: $\Delta \bigcap L_i = \varnothing$ and the origin 0 is a limit point of the set Δ .

Let $i \in \overline{1,n}$, $\Delta \in \Pi_i$ and f be a function defined in some neighborhood of a point $x \in \mathbb{R}^n$. If there exists the limit

$$\lim_{\Delta \ni h \to 0} \frac{f(x+h) - f(x+h(i))}{h_i},$$

then we call its value (i,Δ)-partial derivative of f at point x and denote it by $D_{i,\Delta} f(x)$.

A basis of generalized gradient generating (briefly: basis) we define as an n-tuple $\Delta = (\Delta_1, \cdots, \Delta_n)$, where $\Delta_i \in \Pi_i$ for each $i \in \overline{1,n}$.

If for a basis $\Delta = (\Delta_1, \cdots, \Delta_n)$ a function f has (i, Δ_i)-derivative at point x for every $i \in \overline{1,n}$, then we will say that f has a generalized gradient with respect to basis $\Delta = (\Delta_1, \cdots, \Delta_n)$ (briefly, Δ-gradient) at point x.

The definition of a generalized gradient with respect to a basis represents the direct generalization of definitions of angular gradient and of strong gradient introduced by O. Dzagnidze [Dz1].

Let us briefly review the well-known examples of bases as well as their properties. Along this line, quite exhaustive information is provided by monographs [Dz2] and [Dz3].

For an angle $0 \le \alpha < \pi/2$, let us denote the basis for which

$$\Delta(\alpha)_i = \left\{ h \in \mathbb{R}^n : \frac{\max_{j \ne i} |h_j|}{|h_i|} \le tg\alpha \right\} \quad (i \in \overline{1,n}).$$

It is obvious that $\Delta(0) = (Ox_1 \setminus \{0\}, \cdots, Ox_n \setminus \{0\})$.

Let us define $\Delta(\pi/2)$ as the basis $(\mathbb{R}^n \setminus L_1, \cdots, \mathbb{R}^n \setminus L_n)$.

Let us pick out separately the "initial" - $\Delta(0)$, „intermediate"- $\Delta(\alpha)$ $(0 < \alpha < \pi/2)$ and "final" - $\Delta(\pi/2)$ cases.

Note that: the basis $\Delta(0)$ generates the ordinary gradient(i.e. the condition of existence of ordinary partial derivatives); and the notion of $\Delta(\pi/2)$-gradient coincides with the one of strong gradient.

What else do we know about $\Delta(\alpha)$ bases? Note that first of all we are interested in interrelation between condition of existence of $\Delta(\alpha)$-gradient and condition of differentiability. Along this line, there are known the following interesting results:

1) *The condition of existence* $\Delta(0)$ *-gradient (ordinary gradient) is essentially weaker than condition of differentiability* (Tolstov [To, §4]);
2) *For* $\frac{\pi}{4} \le \alpha < \frac{\pi}{2}$ *the condition of existence of* $\Delta(\alpha)$ *-gradient is equivalent to differentiability condition* (O. Dzagnidze [Dz1]);
3) *The condition of existence of* $\Delta(\pi/2)$ *-gradient (strong gradient) is essentially stronger than the condition of differentiability* (G.G. Oniani [O2], see Theorem 1.2.A).

Thus, in limit cases - $\Delta(0)$ and $\Delta(\pi/2)$ we derive essentially weaker and essentially stronger conditions than differentiability is, respectively; but "the second half" of intermediate cases - $\Delta(\alpha)$ $(\pi/4 \le \alpha < \pi/2)$ bases give the condition equivalent to differentiability.

Let $\Delta = (\Delta_1, ..., \Delta_2)$ be a basis. For a function $f : \mathbb{R}^n \to \mathbb{R}$ by $E_\Delta(f)$ denote the set of all points at which f has $\Delta-$gradient, and by $E_D(f)$ denote the set of all points at which f is differentiable.

Denote also:

$$\|h\| = \max\{|h_1|,...,|h_n|\} \ (h \in \mathbb{R}^n),$$
$$\mathrm{I}(r) = \{h \in \mathbb{R}^n : h \neq 0, \|h\| < r\}, \ (r > 0),$$
$$\mathrm{I}(x,r) = \{y \in \mathbb{R}^n : \|y - x\| < r\} \ (x \in \mathbb{R}^n, r > 0),$$
$$\overline{M} = M \cup \partial M \ (M \subset \mathbb{R}^n).$$

Let us call a basis $\Delta = (\Delta_1,...,\Delta_n)$ regular, if there exist $r > 0$ and $0 < \alpha < \pi/2$ such that $\Delta_1 \cap \mathrm{I}(r) \subset \Delta(\alpha)_1$, . . . , $\Delta_n \cap \mathrm{I}(r) \subset \Delta(\alpha)_n$.

Let us call a basis $\Delta = (\Delta_1,...,\Delta_n)$ complete, if there is exists $r > 0$ such that $\mathrm{I}(r) \subset \Delta_1 \cup ... \cup \Delta_n$.

For a basis $\Delta = (\Delta_1,...,\Delta_n)$ by $\overline{\Delta}$ denote its closure - $\overline{\Delta} = (\overline{\Delta}_1 \setminus L_1,...,\overline{\Delta}_n \setminus L_n)$.

In the second paragraph there is considered the following question: Let $\Delta = (\Delta_1,...,\Delta_n)$ be a basis. When is the condition of existence of Δ-gradient equivalent to differentiability condition? Generally, what kind of interrelation is between the condition of existence of Δ-gradient and differentiability condition?

The theorems given below provide a complete answer to the question for the classes of both arbitrary and continuous functions.

Theorem 2.2.1. *In order that existence of Δ-gradient might imply differentiability (i.e. that the condition $E_\Delta(f) \subset E_D(f)$ might be fulfilled for each function $f : \mathbb{R}^n \to \mathbb{R}$), it is necessary and sufficient that Δ be complete.*

Theorem 2.2.2. *In order that for every continuous function existence of Δ-gradient might imply differentiability (i.e. that the condition $E_\Delta(f) \subset E_D(f)$ might be fulfilled for each function $f : \mathbb{R}^n \to \mathbb{R}$), it is necessary and sufficient that $\overline{\Delta}$ be complete.*

Theorem 2.2.3. *If Δ is regular, then differentiability implies existence of Δ-gradient (i.e. the condition $E_D(f) \subset E_\Delta(f)$ is fulfilled for every function $f : \mathbb{R}^2 \to \mathbb{R}$).*

Theorem 2.2.4. *If Δ is irregular, then differentiability does not imply existence of Δ-gradient, and moreover, there exists a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ such that $E_D(f) \setminus E_\Delta(f) \neq \varnothing$.*

From Theorems 2.2.1-2.2.4 there follow

Corollary 2.2.1. *In order the condition of existence of Δ-gradient to be equivalent to the differentiability condition (i.e. the equality $E_D(f) = E_\Delta(f)$ to be fulfilled for every function $f : \mathbb{R}^n \to \mathbb{R}$) it is necessary and sufficient that Δ be regular and complete.*

Corollary 2.2.2. *In order the condition of existence of Δ-gradient to be equivalent to the differentiability condition in the class of continuous functions (i.e. the equality $E_D(f) = E_\Delta(f)$ to be fulfilled for every continuous function $f : \mathbb{R}^n \to \mathbb{R}$) it is necessary and sufficient that Δ be regular and and $\overline{\Delta}$ be complete.*

In the third paragraph there is considered a question of comparing a condition of existence of generalized gradient with respect to a basis and the differentiability condition on positive measure sets.

According to Theorems 2.2.1 and 2.2.4, if Δ basis is complete and Δ allows anisotropic (tangential) approximation of increment of argument to zero, then the condition of existence of Δ-gradient is stronger than the differentiability condition. Also we know that (see Theorem 1.1.A) if we are dealing with the basis $\Delta(\pi/2)$ (i.e. when anisotropy of increment is not limited), then we derive a considerably stronger condition than differentiability is.

The question arises: which anisotropy parameter does cause creation of the condition, which is essentially stronger than differentiability?

Below we will formulate result, which shows that sufficient property for essential strengthening of differentiability condition is represented by an "anisotropic density"(from the measure standpoint) of a set of increments (basis).

Let us introduce exact definitions.

For an interval $I = I_1 \times \cdots \times I_n$ denote

$$r_i(I) = \frac{\max_{j \neq i} |I_j|}{|I_i|} \qquad \left(i \in \overline{1,n}\right).$$

Let us call a set $E \subset \mathbb{R}^n$ anisotropically dense at point 0 with respect to i -*th* variable, if there exist a number $\alpha > 0$ and a sequence of n - dimensional intervals $(I_k)_{k \in \mathbb{N}}$ with the following properties: $diam\, I_k \to 0$ $(k \to \infty)$; 0 is a center of I_k $(k \in \mathbb{N})$; $r_i(I_k) \to \infty$ $(k \to \infty)$; $\frac{|E \cap I_k|}{|I_k|} \geq \alpha$ $(k \in \mathbb{N})$.

A basis $\Delta = (\Delta_1, \cdots, \Delta_n)$ will be called anisotropically dense, if among its components Δ_i at least one is anisotropically dense at 0 with respect to i -*th* variable.

Theorem 2.3.1. *If a basis* Δ *is anisotropically dense, then there is exists a continuous function* $f : \mathbb{R}^n \to \mathbb{R}$ *such that:*

1) f *is differentiable almost everywhere,*

2) f *almost nowhere has a* Δ *-gradient.*

Corollary 2.3.1. *If a basis* Δ *is complete and anisotropically dense, then the condition of existence of* Δ *-gradient is essentially stronger than the condition of differentiability.*

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